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$N = 2$ de Sitter Super-symmetry Algebra

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Abstract It was shown that $N = 1$ super-symmetry algebra can be constructed in de Sitter space (Pahlavan et al. in Phys Lett. B 627:217–223, 2005), through calculation of charge conjugation in the ambient space notation (Moradi et al. in Phys. Lett. B 613:74, 2005; Phys. Lett. B 658:284, 2008). Calculation of $N = 2$ super-symmetry algebra constitutes the main frame of this paper. $N = 2$ super-symmetry algebra was presented in Pilch et al. (Commun. Math. Phys. 98:105, 1985). In this paper, we obtain an alternative $N = 2$ super-symmetry algebra.

Keywords Super-symmetry · de Sitter space · Algebra

1 Introduction

Recent astrophysical data confirm that our universe could be suitably represented by de Sitter universe. Extensive studies have been dedicated to Q.F.T. in de Sitter universe. Pilch et al concentrated on super-symmetry (susy) in de Sitter universe [4]. Drawing similarities with Minkowskian space they selected a special charge conjugation, which enabled to closed the supper algebra for even value of N. By symmetry consideration of spinor field in de Sitter ambient space notation, a new charge conjugation is obtained [2, 3]. It is shown that $N = 1$ super-symmetry algebra can be constructed in this case [1].

In the present work we have considered $N = 2$ super-symmetry algebra in de Sitter ambient space notation. After introducing the ambient space notation (Sect. 2), the general de Sitter super-algebra is recalled in Sect. 3. In Sect. 4, we have shown that in this case two different algebras do exist; (1) one is the same as previous super-symmetry algebra [4] and (2) the other is an alternative super-symmetry algebra, mentioned but not explored in [4]. Finally, a brief conclusion and an outlook have been discussed in Sect. 5.

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2 Notation

The de Sitter space is an elementary solution of the positive cosmological Einstein equation in the vacuum. It is conveniently seen as a hyperboloid embedded in a five-dimensional Minkowski space

$$X_H = \{x \in \mathbb{R}^5; x^2 = \eta_{\alpha\beta} x^\alpha x^\beta = -H^{-2}\}, \quad \alpha, \beta = 0, 1, 2, 3, 4, \tag{1}$$

where $\eta_{\alpha\beta}$ =diag$(1, -1, -1, -1, -1)$. The de Sitter metrics reads

$$ds^2 = \eta_{\alpha\beta} dx^\alpha dx^\beta \mid_{x^2=-H^{-2}} = g^{dS}_{\mu\nu} dX^\mu dX^\nu, \quad \mu = 0, 1, 2, 3,$$

where the X^μ's are the 4 space-time intrinsic coordinates on dS hyperboloid. Different coordinate systems can be chosen for X^μ. A 10-parameter group $SO_0(1,4)$ is the kinematical group of the de Sitter universe. In the limit $H = 0$, this reduces to the Poincaré group. There are two Casimir operators

$$\begin{aligned} Q^{(1)} &= -\frac{1}{2} M_{\alpha\beta} M^{\alpha\beta}, \\ Q^{(2)} &= -W_\alpha W^\alpha, \quad W_\alpha = -\frac{1}{8} \epsilon_{\alpha\beta\gamma\delta\eta} M^{\beta\gamma} M^{\delta\eta}, \end{aligned} \tag{2}$$

where $\epsilon_{\alpha\beta\gamma\delta\eta}$ is the usual antisymmetrical tensor and the $M_{\alpha\beta}$'s are the infinitesimal generators, which obey the commutation relations

$$[M_{\alpha\beta}, M_{\gamma\delta}] = -i(\eta_{\alpha\gamma} M_{\beta\delta} + \eta_{\beta\delta} M_{\alpha\gamma} - \eta_{\alpha\delta} M_{\beta\gamma} - \eta_{\beta\gamma} M_{\alpha\delta}). \tag{3}$$

$M_{\alpha\beta}$ can be represented as $M_{\alpha\beta} = L_{\alpha\beta} + S_{\alpha\beta}$, where $L_{\alpha\beta} = -i(x_\alpha \partial_\beta - x_\beta \partial_\alpha)$ is the "orbital" part and $S_{\alpha\beta}$ is the "spinorial" part. The form of the $S_{\alpha\beta}$ depends on the spin of the field. For spin $\frac{1}{2}$ field, it can be defined as

$$S_{\alpha\beta} = -\frac{i}{4} [\gamma_\alpha, \gamma_\beta], \tag{4}$$

where the five 4×4 matrices γ^α are the generators of the Clifford algebra based on the metric $\eta_{\alpha\beta}$:

$$\gamma^\alpha \gamma^\beta + \gamma^\beta \gamma^\alpha = 2\eta^{\alpha\beta} \mathbb{1}, \quad \gamma^{\alpha\dagger} = \gamma^0 \gamma^\alpha \gamma^0. \tag{5}$$

An explicit and convenient representation is provided by [5–7]

$$\begin{aligned} &\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \qquad \gamma^4 = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}, \\ &\gamma^1 = \begin{pmatrix} 0 & i\sigma^1 \\ i\sigma^1 & 0 \end{pmatrix}, \qquad \gamma^2 = \begin{pmatrix} 0 & -i\sigma^2 \\ -i\sigma^2 & 0 \end{pmatrix}, \qquad \gamma^3 = \begin{pmatrix} 0 & i\sigma^3 \\ i\sigma^3 & 0 \end{pmatrix}, \end{aligned} \tag{6}$$

where $\mathbb{1}$ is the unit 2×2 matrix and σ^i are the Pauli matrices. This representation had been proved to be useful in analysis of the physical relevance of the group representation [6]. In this representation,

$$\gamma^{\alpha T} = \gamma^4 \gamma^2 \gamma^\alpha \gamma^2 \gamma^4.$$

The spinor wave equation in de Sitter space-time has been originally deduced by Dirac in 1935 [8], and can be obtained from the eigenvalue equation of the second order Casimir operator [5, 6]

$$(-i\not{x}\gamma.\bar{\partial}+2i+\nu)\psi(x)=0, \tag{7}$$

where $\not{x}=\eta_{\alpha\beta}\gamma^\alpha x^\beta$ and $\bar{\partial}_\alpha=\partial_\alpha+H^2x_\alpha x\cdot\partial$. Due to covariance of the de Sitter group, the adjoint spinor is defined as follows [6]:

$$\overline{\psi}(x)\equiv\psi^\dagger(x)\gamma^0\gamma^4. \tag{8}$$

The charge conjugation spinor ψ^c was calculated in the ambient space notation [2, 3]

$$\psi^c=\eta_c C(\gamma^4)^T(\bar{\psi})^T, \tag{9}$$

where η_c is an arbitrary unobservable phase value, generally set to unity. It is easily shown that ψ^c transforms in the same way as ψ [2, 3]

$$\psi'^c(x')=g(\Lambda)\psi^c(x).$$

The wave equation of ψ^c is different from the wave equation of ψ by the sign of the "charge" q and ν [2, 3]. Thus it follows that if ψ describes the motion of a dS-Dirac "particle" with the charge q, ψ^c represents the motion of a dS-Dirac "anti-particle" with the charge $(-q)$. In other words ψ and ψ^c can describe "particle" and "antiparticle" wave functions. ψ and ψ^c are charge conjugation of each other

$$(\psi^c)^c=C\gamma^0\psi^{c*}=C\gamma^0(C\gamma^0\psi)=\psi. \tag{10}$$

In the present framework charge conjugation is an antilinear transformation. In the γ representation (6) we had obtained [2, 3]:

$$\begin{aligned}&C\gamma^0C^{-1}=\gamma^0, \qquad C\gamma^4C^{-1}=-\gamma^4,\\ &C\gamma^1C^{-1}=\gamma^1, \qquad C\gamma^3C^{-1}=\gamma^3, \qquad C\gamma^2C^{-1}=-\gamma^2.\end{aligned} \tag{11}$$

In this representation C anticommutes with γ^2 and γ^4 and commutes with other γ-matrix therefore the simple choice may be taken to be as $C=\gamma^2\gamma^4$, where the following relation is satisfied

$$C=-C^{-1}=-C^T=-C^\dagger. \tag{12}$$

This clearly illustrates the non-singularity of C.

3 Super-symmetry Algebra

Super-symmetry in de Sitter space has been studied by Pilch et al. [4]. Recently, it has been investigated in constant curvature space as well [9, 10]. In this section we have presented the super-symmetry algebra in ambient space notation. It is shown that if the spinor field and the charge conjugation operators are defined in the ambient space notation, a new de Sitter super-algebra can be attained.

In order to extend the de Sitter group, the generators of super-symmetry transformation Q_i^n are introduced. Here i is the spinor index $(i=1,2,3,4)$ and n is the super-symmetry index $n=1,\dots,N$. Q_i^n's are super-algebra spinor generators which transform as de Sitter group spinors. The generators $\tilde{Q}_i^n$ are defined by

$$\tilde{Q}_i = \left(Q^T \gamma^4 C\right)_i = \bar{Q}^c{}_i, \tag{13}$$

where Q^T is the transpose of Q. It can be shown that $\tilde{Q}\gamma^4 Q$ is a scalar field under the de Sitter transformation [2, 3]. For $N \neq 1$, closure of algebra requires extra bosonic generators. These do not necessarily commute with other generators and consequently are not central charges. They are internal symmetry generators [4]. These generators, shown by T_{mn}, commute with de Sitter generators.

Therefore the de Sitter super-algebra in four-dimensional space-time has the following generators:

- $M_{\alpha\beta}$, the generators of de Sitter group,
- the internal group generators T_{nm}, that are defined by the additional condition

$$T_{nm} = -T_{mn}; \quad n,m = 1,2,\dots,N,$$

- the 4-component dS-Dirac spinor generators,

$$Q_i^n, \quad i=1,2,3,4,\ n=1,2,\dots,N.$$

To every generator A, a grade p_a is assigned. For the fermionic generator $p_a=1$, and for the bosonic generator $p_a=0$. The bilinear product operator is defined by

$$[A,B] = -(-1)^{p_a \cdot p_b}[B,A]. \tag{14}$$

The generalized Jacobi identities are

$$(-1)^{p_a\cdot p_c}[A,[B,C]] + (-1)^{p_c\cdot p_b}[C,[A,B]] + (-1)^{p_b\cdot p_a}[B,[C,A]] = 0. \tag{15}$$

Using different generalized Jacobi identities, similar to the method presented by Pilch el al. [4], the full dS-super-algebra can be written in the following form [2, 3]:

$$\begin{aligned}
&[M_{\alpha\beta}, M_{\gamma\delta}] = -i(\eta_{\alpha\gamma}M_{\beta\delta} + \eta_{\beta\delta}M_{\alpha\gamma} - \eta_{\alpha\delta}M_{\beta\gamma} - \eta_{\beta\gamma}M_{\alpha\delta}),\\
&[T_{rl}, T_{pm}] = -i(\omega_{rp}T_{lm} + \omega_{lm}T_{rp} - \omega_{rm}T_{lp} - \omega_{lp}T_{rm}),\\
&[M_{\alpha\beta}, T_{rl}] = 0,\\
&[Q_i^r, M_{\alpha\beta}] = (S_{\alpha\beta}Q^r)_i, \quad [\tilde{Q}_i^r, M_{\alpha\beta}] = -(\tilde{Q}^r S_{\alpha\beta})_i,\\
&[Q_i^r, T^{lp}] = -(\omega^{rl}Q_i^p - \omega^{rp}Q_i^l),\\
&\{Q_i^r, Q_j^l\} = \omega^{rl}\left(S^{\alpha\beta}\gamma^4\gamma^2\right)_{ij} M_{\alpha\beta} + \left(\gamma^4\gamma^2\right)_{ij} T^{rl},
\end{aligned} \tag{16}$$

where $S_{\alpha\beta}$ is a matrix, (4), that satisfying the same algebra as $M_{\alpha\beta}$. It is necessary to obtain matrix ω, which determines the structure of the internal group. In the next section, $N=2$ super-symmetry algebra is considered.

4 $N=2$ Super-symmetry Algebra

For $N=2$, the internal generators T_{mn} are, $T_{11}=T_{22}=0$ and $T_{12}=-T_{21}=T$. There are two possibilities for ω, $\omega=\begin{pmatrix}0&0\\0&0\end{pmatrix}$, and $\omega=\begin{pmatrix}1&0\\0&1\end{pmatrix}$. For the trivial case $\omega=0$, the de Sitter super-symmetry algebra, is determined by the following relations:

$$\begin{aligned}
&[M_{\alpha\beta}, M_{\gamma\delta}]=-i(\eta_{\alpha\gamma}M_{\beta\delta}+\eta_{\beta\delta}M_{\alpha\gamma}-\eta_{\alpha\delta}M_{\beta\gamma}-\eta_{\beta\gamma}M_{\alpha\delta}),\\
&[T_{rl}, T_{pm}]=0, \qquad [M_{\alpha\beta}, T_{rl}]=0, \quad [Q^r_i, T^{lp}]=0,\\
&[Q^r_i, M_{\alpha\beta}]=(S_{\alpha\beta}Q^r)_i, \qquad [\tilde{Q}^r_i, M_{\alpha\beta}]=-(\tilde{Q}^r S_{\alpha\beta})_i,\\
&\{Q^r_i, Q^l_j\}=\left(\gamma^4\gamma^2\right)_{ij}T^{rl}.
\end{aligned} \tag{17}$$

This is proven by the use of generalized Jacobi identities. This is indeed a new super-algebra.

For $\omega=\mathbb{1}$, the de Sitter super-symmetry algebra is

$$\begin{aligned}
&[M_{\alpha\beta}, M_{\gamma\delta}]=-i(\eta_{\alpha\gamma}M_{\beta\delta}+\eta_{\beta\delta}M_{\alpha\gamma}-\eta_{\alpha\delta}M_{\beta\gamma}-\eta_{\beta\gamma}M_{\alpha\delta}),\\
&[T_{rl}, T_{pm}]=-i(\delta_{rp}T_{lm}+\delta_{lm}T_{rp}-\delta_{rm}T_{lp}-\delta_{lp}T_{rm}),\\
&[Q^r_i, M_{\alpha\beta}]=(S_{\alpha\beta}Q^r)_i, \qquad [\tilde{Q}^r_i, M_{\alpha\beta}]=-(\tilde{Q}^r S_{\alpha\beta})_i,\\
&[Q^r_i, T^{lp}]=-(\delta^{rl}Q^p_i-\delta^{rp}Q^l_i), \qquad [M_{\alpha\beta}, T_{rl}]=0,\\
&\{Q^r_i, Q^l_j\}=\delta^{rl}\left(S^{\alpha\beta}\gamma^4\gamma^2\right)_{ij}M_{\alpha\beta}+\left(\gamma^4\gamma^2\right)_{ij}T^{rl}.
\end{aligned} \tag{18}$$

It is interesting to note that if we multiply these relations with γ^4, we obtained exactly the Pilch's result [4]. This is due to the fact that adjoint spinors are defined in two different definition ways:

$$\begin{aligned}
&\overline{\psi}(x)\equiv\psi^\dagger(x)\gamma^0\gamma^4, \quad \text{our definition},\\
&\overline{\psi}(x)\equiv\psi^\dagger(x)\gamma^0, \quad \text{usual definition}.
\end{aligned}$$

This choice of adjoint spinor has been motivated by the covariance of de Sitter group, contrary to the usual Minkowskian analogy [6].

5 Conclusions

A new charge conjugation has been obtained in the quantization process of the spinor field in de Sitter space constructed through ambient space notation [2, 3, 6]. It is shown that an alternative $N=2$ super-symmetry algebra, mentioned but not explored in [4], can be attained by the use of the spinor fields and charge conjugation in the ambient space notation. This result can be used for construction the simplest $N=2$ de Sitter super-gravity Lagrangian, which will be considered in the forthcoming paper.

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